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Resonantly generated internal waves in a contraction

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The near-resonant flow of a stratified fluid through a localized contraction is considered in the long-wavelength weakly nonlinear limit to investigate the transient development of nonlinear internal waves and whether these might lead to local steady hydraulic flows. It is shown that under these circumstances the response of the fluid will fall into one of three categories, the first governed by a forced Korteweg–de Vries equation and the latter two by a variable-coefficient form of this equation. The variable-coefficient equation is discussed using analytical approximations and numerical solutions when the forcing is of the same (positive) and of opposite (negative) polarity to that of free solitary waves in the fluid. For positive and negative forcing, strong and weak resonant regimes will occur near the critical point. In these resonant regimes for positive forcing the flow becomes locally steady within the contraction, while for negative forcing it remains unsteady within the contraction. The boundaries of these resonant regimes are identified in the limits of long and short contractions, and for a number of common stratifications.

1. Introduction

The dynamics of a stratified fluid in a contraction are of considerable oceanographic and engineering importance owing, among many reasons, to the possibility of flow control occurring. Therefore, much of the work on this topic has taken the approach of using steady hydraulic theory to determine the conditions under which these controls develop and the form they take. Hydraulic theory for a stratified fluid was first used by Wood (1968) in considering the selective withdrawal of fluid from an infinite reservoir through a slowly varying contraction; this dealt with both discrete layers and the limiting case of a continuously stratified fluid. Further work on this was performed by Benjamin (1981). Armi & Farmer (1986) considered the case of two discrete layers, which was applied to the flow in the Strait of Gibraltar by Armi & Farmer (1988). The work on continuously stratified fluids has been extended by Armi & Williams (1993), who presented experiments and theory studying the withdrawal through a contraction from an infinite reservoir. Recently Killworth (1992) has proposed a generalized approach for continuously stratified and layered fluids which is applicable to contractions joining reservoirs and to localized contractions in channels. It is this latter case we intend to study here.

Hydraulic theory makes a quasi-horizontal flow hypothesis, and apart from hydraulic shocks assumes that the flow is steady; therefore no waves occur. One would not expect this assumption to be satisfied in practice, as owing to the stratification there will always be unsteady motion, ranging from basin-scale waves down to microscale turbulence. However, these do not necessarily cause the hydraulic theories to be invalid as they are more concerned with the large-scale aspects of the flow. For example, Pierini (1989) demonstrated that the tidal exchange in the Strait of Gibraltar, which is a hydraulic process, caused the radiation of large-amplitude solitary waves in the Mediterranean Sea. Although hydraulics flows can be used as boundary or initial conditions for transient problems, as done by Pierini (1989), hydraulic theory is not well-suited for studying transient problems. Most significantly, it cannot be easily used to determine how steady hydraulic flows develop, or whether a steady flow will evolve from a particular initial condition. The importance of this is that hydraulic theories concentrate on the position and flow conditions at control points, where the long-wave speed of internal wave modes are zero, as demonstrated by Killworth (1992). At these points the resonant forcing of one particular mode will occur, over and above the forcing of other modes. Therefore, it is instructive to consider the near-critical forcing of internal waves in a contraction, as is the purpose here, as this provides one possible evolution path to hydraulic flows which are steady within the vicinity of the contraction. Conversely, it may show that a steady flow would not be expected to evolve within the contraction. The limitation of this approach is that the transient problem can only be studied in the weakly nonlinear limit, and is confined here to the case of a localized contraction in a channel. Our results are of direct relevance to the steady hydraulic theory of Killworth (1992), specifically case (U2) of that paper.

Two recent papers have dealt with specific models of the near-critical unsteady flow of a stratified fluid through a contraction, largely concentrating on three-dimensional aspects of the wave formation. Melville & Macomb (1987) and Tomasson & Melville (1991) both consider interfacial waves in a channel where there is a difference in the width of the contraction in the upper and lower layers. Although the emphasis of these two papers is on three-dimensional waves, their analysis can be used to demonstrate that, except under exceptional conditions, the generation and propagation of two-dimensional waves is described by the forced Korteweg–de Vries (fKdV) equation

$$A_\tau + \Delta A_x + 6AA_x + A_{xxx} = -G_x, \qquad (1.1)$$

where G is proportional to the perturbation in the width of the contraction. This equation occurs in many other cases of the resonant forcing of waves due to a moving disturbance or flow past an obstacle. For example, Akylas (1984) and Wu (1987) consider it in relation to surface waves, Grimshaw & Smyth (1986) for internal waves, Patoine & Warn (1982) and Malanotte-Rizzoli (1984) for Rossby waves and Grimshaw (1990) for inertial waves. For a general review of this equation see Grimshaw (1992). The conditions under which (1.1) was derived by Tomasson & Melville are limited; in §2 a general derivation for the near-critical flow of a stable stratified fluid through a contraction or over a sill will be presented.

The results of Tomasson & Melville (1991) suggest that in the case where there is no difference of width in the upper and lower layers, the contraction will not cause the generation of any resonant waves. Similarly, the derivation of §2 suggests that if the width of the contraction or shear of the oncoming fluid do not vary with height, then no resonant wave generation is caused by the contraction in the limit of the Boussinesq approximation. The flaw in both of these analyses is that they ignore non-Boussinesq effects. In §3 it will be demonstrated that in a straight-sided contraction with no shear, non-Boussinesq effects cause the resonant generation of waves, and that in this case the canonical equation is the variable-coefficient fKdV equation

$$A_\tau + (\Delta A)_x + 6AA_x + A_{xxx} = \gamma\Delta_x. \qquad (1.2)$$

Here $\Delta(x)$ is the perturbation of the velocity from the long-wave speed, which varies in the same manner as the width of the contraction. This forcing due to the variation in width is considered as positive for $\gamma > 0$ and negative for $\gamma < 0$. For positive forcing the waves generated by the contraction will have the same polarity as free solitary waves solutions of (1.2), while for negative forcing the waves will have opposite polarity. The remaining sections are concerned with an analysis of (1.2) using analytical and numerical methods. In §4 the matching technique used by Smyth (1987) to study (1.1) is used to study the case of positive forcing and this is compared to numerical solutions in §5, while in §6 numerical solutions of (1.2) are used to analyse the case of negative forcing. Finally, the resonant regimes for (1.1) and (1.2) and the implications for local steady hydraulic solutions are discussed for a two-layer fluid and for exponentially and linearly stratified fluids.

2. General case

Consider the inviscid non-diffusive flow of a stratified fluid in a duct. A Cartesian coordinate system $h(x, y, z)$ is introduced, where h is the undisturbed height of the free surface above the origin, x is the horizontal coordinate along the duct, y is the transverse coordinate and z is the vertical coordinate, being positive upwards. The density is $\rho_0 \rho(z-\zeta)$ where ζ is the non-dimensional vertical particle displacement, while the buoyancy frequency $N(z)$ is defined as

$$N^2 = -g\rho'(z)/h\rho(z), \tag{2.1}$$

where g is the magnitude of the acceleration due to gravity. The Boussinesq parameter, which measures the strength of the density stratification, is now defined as

$$\beta = hN_0^2/g, \tag{2.2}$$

Where N_0 is a characteristic value of the buoyancy frequency. It is apparent that $N^2 \sim \beta g/h$, and we normalize by defining

$$M(z) = N^2/N_0^2. \tag{2.3}$$

The time variable is $N_0^{-1}t$, while the fluid velocities are $N_0 h(u, v, w)$, the free-surface displacement is $h\eta$ and finally the pressure is written as $\rho_0 gh(p_0(z)+\beta p)$, where $p_0' = -\rho$. The equations of motion describing flow in a duct with side boundaries $y = b_\pm(x, z)$ and base $z = d(x)$ are therefore in dimensionless form

$$\left.\begin{aligned} \rho(z-\zeta)\,\mathrm{d}\boldsymbol{u}/\mathrm{d}t &= -\nabla p + (1/\beta)(\rho(z)-\rho(z-\zeta))\,\boldsymbol{k}, \\ \mathrm{d}\zeta/\mathrm{d}t &= w, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned}\right\} \tag{2.4}$$

with the boundary conditions

$$\left.\begin{aligned} p_0 + \beta p &= 0 \\ w &= e\,\mathrm{d}\eta/\mathrm{d}t \end{aligned}\right\} \quad \text{on} \quad z = 1 + e\eta(x, y, t),$$

$$w = ud_x \quad \text{on} \quad z = d(x),$$

$$v = \boldsymbol{u}\cdot\nabla b_\pm \quad \text{on} \quad y = b_\pm(x, z).$$

The parameter e is 0 for a rigid-lid boundary condition and 1 for a free-surface boundary condition.

The fluid is assumed to have undisturbed velocity $\bar{u}(z)$ and stratification $\rho(z)$. If the combination of these are such that the flow is close to resonance, then it would be expected that any perturbation in the sides or base of the channel will result in the generation of nonlinear waves forced by these perturbations. Here, for simplicity, we are supposing that the perturbations (i.e. $b_\pm$ and d) are introduced for $t \geqslant 0$. The flow then adjusts to these perturbations. An alternative is to suppose that the undisturbed velocity $\bar{u}(z)$ is turned on at $t = 0$. For the ultimate resonant flow the outcome is the same, the difference being in small-amplitude starting transients. To derive a general evolution equation to describe this situation we first assume that the horizontal and time scales are long, and introduce

$$(x, y, b_\pm, t) = \mu^{-1}(x', y', b'_\pm, t'). \tag{2.5}$$

Next, the perturbations in the width and depth are assumed to be small, and can be written as

$$b'_\pm = \pm b_0(1+\epsilon f_\pm), \quad d = \epsilon d'. \tag{2.6}$$

Assuming that the response of the fluid is $O(\alpha)$, we introduce the scaled variables

$$u = \bar{u}'(z) + \alpha u', \quad v = \epsilon v', \quad w = \mu\alpha w', \quad p = \alpha p', \quad \zeta = \alpha\zeta', \quad \eta = \beta\alpha\eta'. \tag{2.7}$$

An alternative approach for the transverse variables is to assume that y, $b_\pm = O(1)$ and $v = O(\mu\epsilon)$. This leads to a slightly simpler derivation of the following equations, but does not have the same generality.

The usual balance for flow near resonance is made, and so $\alpha = \mu^2 = \epsilon^{1/2}$. For convenience the primes are now dropped. It is assumed that the basic state $\bar{u}(z)$, $\rho(z)$ is such that there is a linear long-wave mode whose phase speed is $O(\epsilon^{1/2})$. Hence we let

$$\bar{u}(z) = -c(z) + \epsilon^{1/2}\Delta + O(\epsilon), \quad \tau = \epsilon^{1/2} t, \tag{2.8}$$

where $c(z)$ is such that the linear long-wave modal function $\phi(z)$ satisfies the boundary-value problem

$$(\rho c^2 \phi_z)_z + \rho M \phi = 0, \tag{2.9}$$

with

$$\phi = \begin{cases} e\beta c^2 \phi_z & \text{on} \quad z = 1, \\ 0 & \text{on} \quad z = 0. \end{cases}$$

The equations of motion are now

$$\left.\begin{aligned} &(\rho + \beta\epsilon^{1/2}\rho M\zeta)(-cu_x - c_z w + \epsilon^{1/2}\,\mathrm{d}u/\mathrm{d}\tau) + p_x = O(\epsilon), \\ &p_y = O(\epsilon^{1/2}), \\ &p_z + \rho M\zeta - \epsilon^{1/2}(\rho c w_x + \tfrac{1}{2}(\rho M)_z \zeta^2) = O(\epsilon), \\ &c\zeta_x + w - \epsilon^{1/2}\,\mathrm{d}\zeta/\mathrm{d}\tau = O(\epsilon), \\ &u_x + w_z + \epsilon^{1/2} v_y = 0, \end{aligned}\right\} \tag{2.10}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\partial}{\partial\tau} + (\Delta + u)\frac{\partial}{\partial x} + w\frac{\partial}{\partial z},$$

and with the boundary conditions

$$w + \frac{e\beta c p_x}{\rho} + \epsilon^{1/2}\left[\frac{e\beta w_z p}{\rho} + \frac{e\beta^2}{2\rho}\left(\frac{cp^2}{\rho}\right)_{xz} - e\beta\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{p}{\rho}\right)\right] = O(\epsilon) \quad \text{on} \quad z = 1,$$

$$w = -\epsilon^{1/2} d_x c + O(\epsilon) \quad \text{on} \quad z = 0,$$

$$v = \mp b_0 c(f_\pm)_x + O(\epsilon^{1/2}) \quad \text{on} \quad y = \pm b_0.$$

We seek a perturbation solution of (2.10) by introducing expansions of the form

$$g = g^{(0)} + \epsilon^{1/2} g^{(1)} + O(\epsilon) \tag{2.11}$$

for u, v, w, p and ζ. At zeroth order it can be shown that (2.10) has the solution

$$u^{(0)} = A(x,\tau)(c\phi(z))_z, \quad w^{(0)} = -cA_x\phi, \quad \zeta^{(0)} = A\phi, \quad p^{(0)} = c^2A\phi_z, \tag{2.12}$$

where ϕ satisfies the boundary-value problem (2.9).

Taking the expansion to first order, we define

$$\hat{\zeta}^{(1)} = \frac{1}{2b_0}\int_{-b_0}^{b_0} \zeta^{(1)}\,\mathrm{d}z, \quad f = \tfrac{1}{2}(f_+ + f_-). \tag{2.13}$$

Then it can be shown that $\hat{\zeta}^{(1)}$ satisfies the non-homogeneous boundary-value problem

$$(\rho c^2\hat{\zeta}^{(1)}_{xz})_z + \rho M\hat{\zeta}^{(1)} = 2(A_\tau + \Delta A_x)(\rho c\phi_z)_z - A_{xxx}\rho c^2\phi - (\rho c^2 f_x)_z$$
$$+ AA_x(2(\rho c^2(\phi_z^2 - \phi\phi_{zz}))_z + (\rho c^2)_z\phi_z^2), \tag{2.14}$$

with

$$\hat{\zeta}^{(1)}_x - e\beta c^2\hat{\zeta}^{(1)}_{xz} = e\beta(c^2 f_x - 2(A_\tau + \Delta A_x)c\phi_z - AA_x c^2(3\phi_z^2 - 2\phi\phi_{zz})) \quad \text{on} \quad z = 1,$$
$$\hat{\zeta}^{(1)}_x = d_x \quad \text{on} \quad z = 0.$$

The compatibility condition for (2.14) is obtained by multiplying by ϕ and integrating over the interval $z = (0, 1)$. It is then found that A must satisfy the forced KdV equation

$$A_\tau + \Delta A_x + rAA_x + sA_{xxx} = -G_x, \tag{2.15}$$

where

$$r = \frac{3\int_0^1 \rho c^2\phi_z^3\,\mathrm{d}z}{2\int_0^1 \rho c\phi_z^2\,\mathrm{d}z}, \quad s = \frac{\int_0^1 \rho c^2\phi^2\,\mathrm{d}z}{2\int_0^1 \rho c\phi_z^2\,\mathrm{d}z}, \quad G = \frac{d[\rho c^2\phi_z]_{z=0} - \int_0^1 \rho c^2 f\phi_z\,\mathrm{d}z}{2\int_0^1 \rho c\phi_z^2\,\mathrm{d}z}. \tag{2.16}$$

Equation (2.15) can be transformed to (1.1) by introducing

$$\tau^* = s\tau, \quad \Delta^* = \Delta/s, \quad A^* = rA/6s, \quad G^* = rG/6s. \tag{2.17}$$

Dropping the asterisks, the new variables will now satisfy (1.1). The behaviour of (1.1) will not be studied here, as extensive details of the solutions are presented in the literature, e.g. Grimshaw & Smyth (1986), Smyth (1987).

When $f = 0$ and $c_z = 0$ (2.15) reduces to that derived by Grimshaw & Smyth (1986). In that case the mechanism that generates the forced wave is that the bottom topography causes the isopycnals to be displaced, the fluid then acts to correct this displacement and hence a forced wave forms. When $d = 0$ a similar mechanism can be shown to hold. The presence of the contraction induces a transverse velocity, which in general varies with height z, hence a vertical velocity is induced and the isopycnals are displaced, resulting in a resonant wave being generated. If however in the Boussinesq limit $\beta \rightarrow 0$, f and c are both independent of z (i.e. $f_z = c_z = 0$) then the transverse velocity does not vary with height, and the induced vertical velocity cannot be a function of z and hence, to satisfy the boundary condition, must be zero. Therefore, the isopycnals are not displaced and no resonant wave would be expected to be generated. Indeed, when $d = 0$ and $(c^2f)_z = 0$ then $G = 0$. This would indicate that a rescaling is necessary, which we examine in the next section.

Note that if $r = O(\beta)$ the analysis also fails, which includes stratifications close to uniform. In this case a finite-amplitude analysis must be undertaken, which will result in an equation of the form derived by Grimshaw & Yi (1991).

3. The Boussinesq limit

3.1. *General stratification*

From this point it will be assumed that the duct is flat bottomed (i.e. $d = 0$) and that both the side boundaries and horizontal velocity are constant with height (i.e. $f_z = c_z = 0$). The derivation of the previous section would suggest that in this case for the Boussinesq limit $\beta \to 0$ no forced wave will be formed near resonance. The reason for this is that in this joint limit there can exist a steady flow through the contraction which is independent of height. If we denote this steady flow by $\bar{u}$, $\epsilon\bar{v}$, $\epsilon\bar{p}$ with $\bar{w} = \bar{\zeta} = \bar{\eta} = 0$, then with $\beta = 0$,

$$\left.\begin{aligned} \bar{u}\bar{u}_x + \epsilon\bar{v}\bar{u}_y &= -\epsilon\bar{p}_x, \\ \bar{u}\bar{v}_x + \epsilon\bar{v}\bar{v}_y &= -\bar{p}_y, \\ \bar{u}_x + \epsilon\bar{v}_y &= 0, \end{aligned}\right\} \tag{3.1}$$

with $$\bar{v} = \pm\bar{u}b_0(f_\pm)_x \quad \text{on} \quad y = \pm b_0(1+\epsilon f_\pm).$$

In this limit $\bar{u}$, $\bar{v}$ and $\bar{p}$ are dependent only on x, y; therefore from (3.1) we find that

$$\frac{1}{2b_0}\int_{-b_0}^{b_0} \bar{u}\,\mathrm{d}y = U(1-\epsilon f + O(\epsilon^2)), \tag{3.2}$$

where U is a constant. If the $O(\beta)$ Boussinesq correction is retained, then p is replaced by $p/\rho(z)$ and an $O(\epsilon\beta)$ z-dependent term is introduced into the x-momentum equation. At $O(\epsilon\beta)$ the assumption that there exists a steady flow through the contraction now fails. From (2.15) and (2.16) it is apparent that, in effect, an $O(\epsilon\beta)$ forcing term is provided by the contraction. The undisturbed steady flow $\bar{u}$ must now adjust by the generation of nonlinear waves. At resonance, the response of the fluid is $O(\alpha)$ where $\alpha = O((\epsilon\beta)^{1/2})$, which suggests the obvious choice for the Boussinesq parameter $\beta = O(\epsilon)$ and hence $\alpha = O(\epsilon)$. Note that if either f_z or c_z are $O(\epsilon)$ a similar response will occur; however, as our interest is primarily with non-Boussinesq effects we will assume that $f_z = c_z = 0$.

To determine the evolution equation describing the resonant generation of waves in this limit, the expressions (2.5) and (2.6) are again introduced. However, (2.7) is now replaced by

$$u = \bar{u}' + \alpha u', \quad v = \epsilon(\bar{v}' + \alpha v'), \quad w = \mu\alpha w', \quad p = \epsilon\bar{p}' + \alpha p', \quad \zeta = \alpha\zeta', \quad \eta = \beta\alpha\eta', \tag{3.3}$$

where $\bar{u}'$, $\bar{v}'$ and $\bar{p}'$ satisfy (3.1). Following the arguments of the preceding paragraph we put $\beta = \sigma\epsilon$ and $\alpha = \mu^2 = \epsilon$. Primes are once again dropped and as the flow is assumed to be near resonance, we introduce

$$\bar{u} = -c + \epsilon\delta(x, y), \quad \tau = \epsilon t, \tag{3.4}$$

where c is now a constant, such that the modal function $\phi(z)$ satisfies the boundary-value problem

$$\phi_{zz} + \frac{M}{c^2}\phi = 0, \tag{3.5}$$

with $$\phi = 0 \quad \text{on} \quad z = 0, 1.$$

Note that $U = -c$ in (3.2) and thus we can show that

$$\hat{\delta}_x = \frac{1}{2b_0}\int_{-b_0}^{b_0} \delta_x\,\mathrm{d}y = cf_x. \tag{3.6}$$

The material derivative is defined as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\partial}{\partial \tau} + (\delta + u)\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}, \tag{3.7}$$

and therefore the equations of motion (2.4) become

$$\left.\begin{aligned} &u_{xz} + M\zeta_x + \epsilon\{-(\mathrm{d}u/\mathrm{d}\tau)_z - \bar{v}u_{yz} - \delta_x u_z - cw_{xx} - M_z\zeta\zeta_x - \sigma c M(u+\delta)_x\} = O(\epsilon^2),\\ &u_y = O(\epsilon),\\ &w + c\zeta_x - \epsilon((\mathrm{d}\zeta/\mathrm{d}\tau) + \bar{v}\zeta_y) = O(\epsilon^2),\\ &u_x + w_z + \epsilon v_y = 0, \end{aligned}\right\} \tag{3.8}$$

with the boundary conditions

$$w = \begin{cases} -e\sigma\epsilon c^2(u+\delta)_x + O(\epsilon^2) & \text{on} \quad z = 1,\\ 0 & \text{on} \quad z = 0, \end{cases}$$

$$v = \mp b_0 c(f_\pm)_x + O(\epsilon) \quad \text{on} \quad y = \pm b_0.$$

To solve (3.8) a perturbation series of the form

$$g = g^{(0)} + \epsilon g^{(1)} O(\epsilon^2) \tag{3.9}$$

is introduced for u, v, w and ζ. At zeroth order the solution of (3.8) is

$$u^{(0)} = cA(x,\tau)\,\phi_z(z), \quad w^{(0)} = -cA_x\phi, \quad \zeta^{(0)} = A\phi, \tag{3.10}$$

where ϕ and c satisfy (3.5).

At first order $\hat{\zeta}^{(1)}$ satisfies a non-homogeneous boundary-value problem of similar form to (2.14), where the operator on the left-hand side is now the same as for (3.5) and the forcing terms on the right-hand side are similar in structure to (2.14). As in §2 the compatibility condition for $\hat{\zeta}^{(1)}$ is found by multiplying this equation by ϕ and integrating over the interval $z = (0, 1)$; thus it can be shown that A must satisfy the forced KdV equation

$$A_\tau + (\Delta A)_x + rAA_x + sA_{xxx} = \gamma\Delta_x, \tag{3.11}$$

where

$$\Delta = \hat{\delta} + \frac{\sigma c^3(\frac{1}{2}[\phi_z^2]_0^1 - e[\phi_z^2]_{z=1})}{2\int_0^1 \phi_z^2\,\mathrm{d}z}, \quad r = \frac{3c\int_0^1 \phi_z^3\,\mathrm{d}z}{2\int_0^1 \phi_z^2\,\mathrm{d}z}, \quad s = \frac{c\int_0^1 \phi^2\,\mathrm{d}z}{2\int_0^1 \phi_z^2\,\mathrm{d}z}, \quad \gamma = \frac{\delta c^2(e[\phi_z]_{z=1} - [\phi_z]_0^1)}{2\int_0^1 \phi_z^2\,\mathrm{d}z}. \tag{3.12}$$

Note that the forcing term $\gamma\Delta_x$ on the right-hand side of (3.11) agrees with $-G_x$ in (2.15) in the present limit if $f_z = c_z = 0$ and $\beta \to 0$. The new feature here is that the detuning parameter $\Delta = \Delta_0 + cf$ now varies with x. Equation (3.11) can be converted to the canonical form, (1.2), by introducing the variables

$$\tau^* = s\tau, \quad \Delta^* = \Delta/s, \quad A^* = rA/(6s), \quad \gamma^* = r\gamma/(6s). \tag{3.13}$$

Dropping the asterisks, the new variables now satisfy (1.2).

By matching the nonlinear and forcing terms of (3.11), it can be seen that the magnitude of the amplitude will be

$$A \sim |\gamma/r|^{1/2}. \tag{3.14}$$

If $\sigma \to 0$ then $A \to 0$ and the forced wave can be neglected. In the opposite limit, if $r \to 0$ then $A \to \infty$ and therefore this derivation breaks down. This is of particular importance for the case of uniform stratification, which we consider next.

3.2. *Uniform stratification*

The term 'uniform stratification' denotes that the buoyancy frequency is constant and consequently $M = 1$. In the Boussinesq limit $\beta \to 0$, we can show that for uniform stratification $r = 0$ and thus the amplitude A will increase without bounds with increasing time. For stratifications which are approximately uniform $r = O(\beta)$ and consequently $A = O(\beta^{-1/2})$. The derivation in the previous section for general stratification used the choice $\alpha = \epsilon$, which since $\gamma = O(\sigma)$, is more correctly $\alpha = (\beta\epsilon)^{1/2}$. Therefore for uniform stratifications it is clear that $\alpha = O(\epsilon^{1/2})$. The requirement that linear and nonlinear advection are of the same order of magnitude then gives that $\beta = O(\epsilon^{1/2})$. An alternative choice which balances linear and nonlinear advection is $\alpha = O(1)$ and $\beta = O(\epsilon)$, which results in a finite-amplitude evolution equation similar to that derived by Grimshaw & Yi (1991); this will be the subject of further study.

Again the expressions (2.5), (2.6) and (3.3) are introduced, with the choice of parameters $\alpha = \mu = \epsilon^{1/2}$ and $\beta = \sigma\epsilon^{1/2}$. Further, we assume that the fluid has approximately uniform stratification and let

$$M(z) = 1 + \beta\chi(z). \tag{3.15}$$

The long-wave speed must then be of the form

$$c = c_0 + \beta c_1, \tag{3.16}$$

where

$$c_0 = 1/(n\pi), \tag{3.17}$$

and the Boussinesq correction term c_1 is unknown at this point.

In this derivation terms to $O(\alpha^2)$ must be retained, as the evolution equation will be determined at this order. Therefore the equations of motion are

$$\left.\begin{aligned} & c_0 u_{xz} + \zeta_x - \epsilon^{1/2}((uu_x + wu_z)_z - \sigma\chi\zeta_x + \sigma c_0 u_x - \sigma c_1 u_{xz}) \\ & \quad - \epsilon((u_\tau + (\delta u)_x + \bar{v}u_y)_z + \sigma^2 c_1 u_x + c_0 w_{xx} \\ & \quad + \sigma\chi_z \zeta\zeta_x + \sigma c_0(\sigma\chi - \zeta_z) u_x - \sigma(uu_x + wu_z) + \sigma c_0 \delta_x) = O(\epsilon^{3/2}), \\ & u_y = O(\epsilon), \\ & w + c_0 \zeta_x - \epsilon^{1/2}(u\zeta_x + w\zeta_z - \sigma c_1 \zeta_x) - \epsilon(\zeta_\tau + \delta\zeta_x + \bar{v}\zeta_y) = O(\epsilon^{3/2}), \\ & u_x + w_z + \epsilon v_y = 0, \end{aligned}\right\} \tag{3.18}$$

with the boundary conditions

$$\begin{aligned} & w + e\sigma\epsilon^{1/2} c_0^2 u_x + e\sigma\epsilon c_0(uw_z - 2uu_x + 2\sigma c_1 u_x + c_0 \delta_x) = O(\epsilon^{3/2}) \quad \text{on} \quad z = 1, \\ & w = 0 \quad \text{on} \quad z = 0, \\ & v = \pm b_0(f_\pm)_x u + O(\epsilon) \quad \text{on} \quad y = \pm b_0. \end{aligned}$$

The form of (3.18) suggests a perturbation solution in $\epsilon^{1/2}$, and so expansions of the form (2.11) are introduced for u, v, w and ζ. At zeroth order the solution of (3.18) is

$$u^{(0)} = c_0 A(x,\tau)\phi_z, \quad w^{(0)} = -c_0 A_x \phi, \quad \zeta^{(0)} = A\phi, \tag{3.19}$$

where

$$\phi = \sin n\pi z. \tag{3.20}$$

At first order it can be shown that the solution of (3.18) is

$$u^{(1)} = \sigma A(c_0 \varphi_z + c_1 \phi_z), \quad w^{(1)} = -\sigma A_x(c_0 \varphi + c_1 \phi_z), \quad \zeta^{(1)} = \sigma A\varphi, \tag{3.21}$$

where, defining

$$g(z) = c_0^2\phi_z + \phi\left(\frac{2c_1}{c_0} - \chi\right), \tag{3.22}$$

φ is given by

$$\varphi = \phi\int_0^z g(\xi)\,\phi_\xi(\xi)\,\mathrm{d}\xi - \phi_z\int_0^z g(\xi)\,\phi(\xi)\,\mathrm{d}\xi. \tag{3.23}$$

At second order, proceeding in the same manner as in §§2 and 3.1, it can be shown that the compatibility condition for $\hat{\zeta}^{(2)}$ is that A must satisfy the forced KdV equation, (3.11), where in this case

$$\left.\begin{aligned} \Delta &= \hat{\delta} + O(\sigma^2), \quad r = \sigma\left(\tfrac{1}{3}c_0^2(1-(-1)^n) - 2c_0\int_0^1 \chi_z\,\phi^3\,\mathrm{d}z + 3ec_0^2(-1)^n\right), \\ s &= \tfrac{1}{2}c_0^3, \quad \gamma = \sigma c_0^3(e(-1)^n + 1 - (-1)^n). \end{aligned}\right\} \tag{3.24}$$

4. Solutions for positive forcing

In the following three sections, it is assumed that Δ is of the form

$$\Delta = \Delta_0 - \Delta_1\,\theta(x/x_a), \tag{4.1}$$

where $\Delta_1 > 0$ and θ satisfies $0 \leqslant \theta \leqslant 1$, $\theta(0) = 1$, $\theta \to 0$ as $|x| \to \infty$. Therefore, the minimum velocity in the contraction is $\Delta(0) = \Delta_0 - \Delta_1$, and is denoted as Δ_m. The first parameter, Δ_0, represents a detuning parameter, while the second, Δ_1, represents the strength of the contraction.

When $\gamma > 0$ solutions of (1.2) can be constructed using the method of Smyth (1987) to solve (1.1) for positive forcing. This involves dividing the domain into an inner region and an outer region. In the inner region, steady solutions to (1.2) which are non-zero as $x \to \pm\infty$ are constructed; they can be found explicitly for the limiting cases of long and short contractions. In the outer region, where variable velocity is no longer important, the governing equation is the KdV equation. In this region modulation solutions of the KdV equation are constructed and then matched to the inner solution.

4.1. *Inner steady solution*

When $x_a \gg 1$, Grimshaw & Smyth (1986) demonstrated that as a first approximation for the steady part of the solution to (1.1) the dispersive term can be ignored. Invoking the same limit for (1.2) and assuming that the solution is steady gives

$$(\Delta A)_x + 6AA_x = \gamma\Delta_x. \tag{4.2}$$

At $x = 0$, since $\Delta_x = 0$, then either $6A = -\Delta$ or $A_x = 0$. The latter case can be ignored as this can only lead to symmetric non-resonant solutions of (4.2). Therefore, for resonant solutions of (4.2), the amplitude at the throat of the contraction must satisfy $6A = -\Delta_m$. Hence, (4.2) can be integrated to give

$$6A = -\Delta \pm (\Delta^2 - \Delta_m^2 + 12\gamma(\Delta - \Delta_m))^{1/2}. \tag{4.3}$$

As Δ can be negative, A will only have real solutions if $\Delta_m \geqslant -6\gamma$. The solutions as $x \to \pm\infty$ are denoted respectively as $A_\pm$, and satisfy $A_+ > A_-$. Therefore, the resonant solution of (4.2) is

$$6A = -\Delta + \operatorname{sgn} x(\Delta^2 - \Delta_m^2 + 12\gamma(\Delta - \Delta_m))^{1/2}. \tag{4.4}$$

The outer limits of this inner solution are

$$6A_{\pm} = -\Delta_0 \pm (\Delta_0^2 - \Delta_m^2 + 12\gamma(\Delta_0 - \Delta_m))^{1/2}. \tag{4.5}$$

In this hydraulic approximation, these solutions $A_{\pm}$ are terminated by shocks propagating with speeds $V_{\pm} = \Delta_0 + 3A_{\pm}$. In the next subsection we replace these shocks with modulated wavetrains, but here we note that since $V_+ > 0$ and $V_- < 0$ for resonant flow, it follows that $\Delta_m^2 < 12\gamma\Delta_1$.

For short contractions, where $x_a \to 0$, the velocity can be written as

$$\Delta = \Delta_0 - \Delta_1 D\delta(x), \tag{4.6}$$

where

$$D = x_a \int_{-\infty}^{\infty} \theta(x)\,\mathrm{d}x. \tag{4.7}$$

For example, if $\theta(x) = \operatorname{sech}^2 x$ then $D = 2x_a$. In this case A satisfies the KdV equation with a discontinuity at $x = 0$. Again, steady solutions in the inner region are sought. It can be shown using a similar approach to Grimshaw & Smyth (1986), that the steady solution with unequal limits as $x \to \pm\infty$ is

$$A_s = \begin{cases} A_+, & x > 0, \\ A_- + 2k^2 \operatorname{sech}^2 k(x - x_0), & x < 0, \end{cases} \tag{4.8}$$

where k is the solution of

$$16k^3 = 3^{1/2}\Delta_1 D(6\gamma + \Delta_0 - 4k^2), \tag{4.9}$$

and

$$x_0 = \frac{1}{2k}\ln(2 + 3^{1/2}), \quad 6A_{\pm} = -\Delta_0 \pm 4k^2. \tag{4.10}$$

4.2. *Outer modulated wave solutions*

The steady solutions have non-zero limits upstream and downstream of the contraction, which fail to satisfy the boundary conditions that $A \to 0$ as $|x| \to \infty$. Travelling wave solutions must be introduced so that these boundary conditions are satisfied. These solutions, which are constructed following Smyth (1987), are based on the modulation equations of Whitham (1965, 1974). The modulation theory assumes that the solution can be written as a slowly varying cnoidal wave:

$$A = b + 2a\left\{\operatorname{cn}^2 \kappa(x - (U + \Delta_0)\tau) + \frac{1-m}{m} - \frac{E}{mK}\right\}, \tag{4.11}$$

where

$$\kappa = \left(\frac{a}{m}\right)^{1/2}, \quad U = 6\left(b + 2a\left(\frac{2-m}{3m} - \frac{E}{mK}\right)\right),$$

m is the modulus of the function cn, while $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind respectively. The modulation theory can then be used to obtain a set of characteristic equations for b, a and m.

Upstream (i.e. $x > 0$) of the inner solution, the appropriate solution of the modulation equations is

$$\left.\begin{aligned} b &= \alpha(m - 1 + 2E/K) \\ a &= \alpha m \\ \frac{x}{\tau} &= \Delta_0 + 2\alpha\left(1 + m - \frac{2m(1-m)K}{E - (1-m)K}\right) \end{aligned}\right\} \text{ on } \quad m_0 \leqslant m \leqslant 1. \tag{4.12}$$

The solution has two unknowns, α and m_0; these are found by matching to the inner solution. As shown by Smyth (1987) the appropriate conditions are that $b \rightarrow A_+$ and $m \rightarrow m_0$ as $x \rightarrow 0$. Hence

$$\alpha = A_+(m_0 - 1 + 2E_0/K_0)^{-1}, \tag{4.13}$$

where K_0, E_0 are respectively $K(m_0)$, $E(m_0)$, and m_0 is the solution of

$$\Delta_0\left(m_0 - 1 + \frac{2E_0}{K_0}\right) + 2A_+\left(1 + m_0 - \frac{2m_0(1-m_0)\,K_0}{E_0 - (1-m_0)\,K_0}\right) = 0. \tag{4.14}$$

The solution therefore occupies the region

$$0 \leqslant x/\tau \leqslant \Delta_0 + 4A_+(m_0 - 1 + 2E_0/K_0)^{-1}. \tag{4.15}$$

Note that as $m \rightarrow 1$, the solution (4.11) approaches a solitary wave of amplitude 2α:

$$A = 2\alpha\,\mathrm{sech}^2\,\alpha^{1/2}(x - (4\alpha + \Delta_0)\,\tau), \tag{4.16}$$

travelling at exactly the same speed as this edge of the modulation solution.

Downstream (i.e. $x < 0$) of the inner solution, the appropriate solution is

$$\left.\begin{aligned} b &= \mu(2 - m - 2E/K) \\ a &= -\mu m \\ \frac{x}{\tau} &= \Delta_0 + 2\mu\left(2 - m + \frac{2m(1-m)\,K}{E - (1-m)\,K}\right) \end{aligned}\right\} \text{ on } \quad 0 \leqslant m \leqslant 1. \tag{4.17}$$

To find μ, the solution is again matched to the inner solution. Now as $m \rightarrow 1$, $b \rightarrow A_-$ and so

$$\mu = A_-. \tag{4.18}$$

Therefore the downstream solution occupies the region

$$\Delta_0 + 12A_- \leqslant x/\tau \leqslant \Delta_0 + 2A_-, \tag{4.19}$$

corresponding to the limits $m \rightarrow 0$ and $m \rightarrow 1$ respectively. Note that the solitary-wave end of the wavetrain is now closest to the contraction.

From the inequality (4.19) it is apparent that the downstream solution breaks down if $2A_- > -\Delta_0$, which is equivalent to $6A_+ < \Delta_0$. When $6A_+ = \Delta_0$, from (4.14) it can be shown that $m_0 = 0$ and consequently $\alpha = A_+$. Hence, if $6A_+ < \Delta_0$, the upstream wavetrain detaches from the inner region and occupies the whole zone $0 \leqslant m \leqslant 1$, so that $m_0 = 0$. We find that

$$\alpha = A_+, \tag{4.20}$$

and

$$\Delta_0 - 6A_+ \leqslant x/\tau \leqslant \Delta_0 + 4A_+. \tag{4.21}$$

This is terminated at $m_0 = 0$ by a mean level A_+, corresponding to the upstream limit of the inner solution.

Downstream, the wavetrain is now attached to the inner region and forms a steady lee wavetrain with modulus $m = m_s$ and velocity $U = -\Delta_0$. Therefore, using (4.11), μ is given by

$$\mu = -\Delta_0/[2(2 - m_s)]. \tag{4.22}$$

There is now a steady wavetrain downstream of the contraction and a steady elevation upstream of the contraction. The forced KdV equation can be integrated over this region, to give

$$\Delta A + 3A^2 + A_{xx} = \Delta_0 A_+ + 3A_+^2 - \gamma(\Delta_0 - \Delta). \tag{4.23}$$

The modulus of the cnoidal wave can be found from this equation, and must satisfy

$$(\Delta_0 m_s)^2 = 4A_+(\Delta_0+3A_+)(m_s-2)^2. \tag{4.24}$$

Hence, if A_+ is known, the modulus can be found. In the case of long contractions, A_+ is given by (4.5), as the solution will reach its asymptotic downstream value before the formation of the wavetrain. However, for short contractions this is no longer so, as the cnoidal wave forms immediately downstream of the contraction and the steady solution is

$$A_s = \begin{cases} A_+, & x>0, \\ \mu m_s(1-2\,\mathrm{cn}^2\,\kappa(x-x_0)), & x<0, \end{cases} \tag{4.25}$$

rather than (4.8). Using continuity conditions across the singularity we can show that

$$\left.\begin{aligned} &\mathrm{cn}^2\,\kappa x_0 = \tfrac{1}{2}(1+2A_+(2-m_s)/(\Delta_0 m_s)), \\ &\frac{8\Delta_0 A_+(\Delta_0+3A_+)}{2-m_s}\mathrm{cn}^2\,\kappa x_0(1-m_s+m_s\,\mathrm{cn}^2\,\kappa x_0)(1-\mathrm{cn}^2\,\kappa x_0) = (\Delta_1 D(A_+-\gamma))^2. \end{aligned}\right\} \tag{4.26}$$

This pair of equations, together with (4.24), can be written explicitly in terms of A_+ or m_s and then solved.

The lee wavetrain will occupy the region

$$-\frac{\Delta_0 2m_s(1-m_s)K_s}{(2-m_s)(E_s-(1-m_s)K_s)} \leqslant \frac{x}{\tau} \leqslant 0, \tag{4.27}$$

where K_s, E_s are respectively $K(m_s)$, $E(m_s)$. A partial wavetrain with an identical value of μ terminates the lee wavetrain. This has modulus in the range

$$0 \leqslant m \leqslant m_s, \tag{4.28}$$

and therefore occupies the region

$$\Delta_0+12\mu \leqslant \frac{x}{\tau} \leqslant -\frac{\Delta_0 2m_s(1-m_s)K_s}{(2-m_s)(E_s-(1-m_s)K_s)}. \tag{4.29}$$

This approach contrasts with that of Smyth (1987), who assumed that the mean height of the cnoidal wavetrain is A_- and therefore m_s is the solution of

$$\Delta_0(2-m_s-2E_s/K_s)+2A_-(2-m_s) = 0. \tag{4.30}$$

Summarizing, these results can be divided into two subregimes. The first of these is the 'strong' resonant regime and occurs when

$$6A_+ > \Delta_0 \quad \text{with} \quad A_- < 0 < A_+. \tag{4.31}$$

In this case a partial wavetrain forms immediately upstream of the inner solution with modulus in the range $m_0 \leqslant m \leqslant 1$, where m_0 is the solution of (4.14). Downstream of the contraction a full wavetrain forms which acts as a transition between the downstream depression of the inner solution and the downstream zero level. This wavetrain has modulus $0 \leqslant m \leqslant 1$. In the 'weak' resonant regime, or lee wave regime,

$$0 < 6A_+ < \Delta_0. \tag{4.32}$$

The upstream wavetrain is now detached from the inner solution, with $m_0 = 0$. This wavetrain is terminated by the upstream limit of the inner solution A_+. Downstream, the wavetrain is attached to the inner solution and a stationary lee wavetrain forms with modulus m_s, given by the solution of (4.24) and (4.26). This is terminated by a partial wavetrain with modulus in the range $0 \leqslant m \leqslant m_s$.

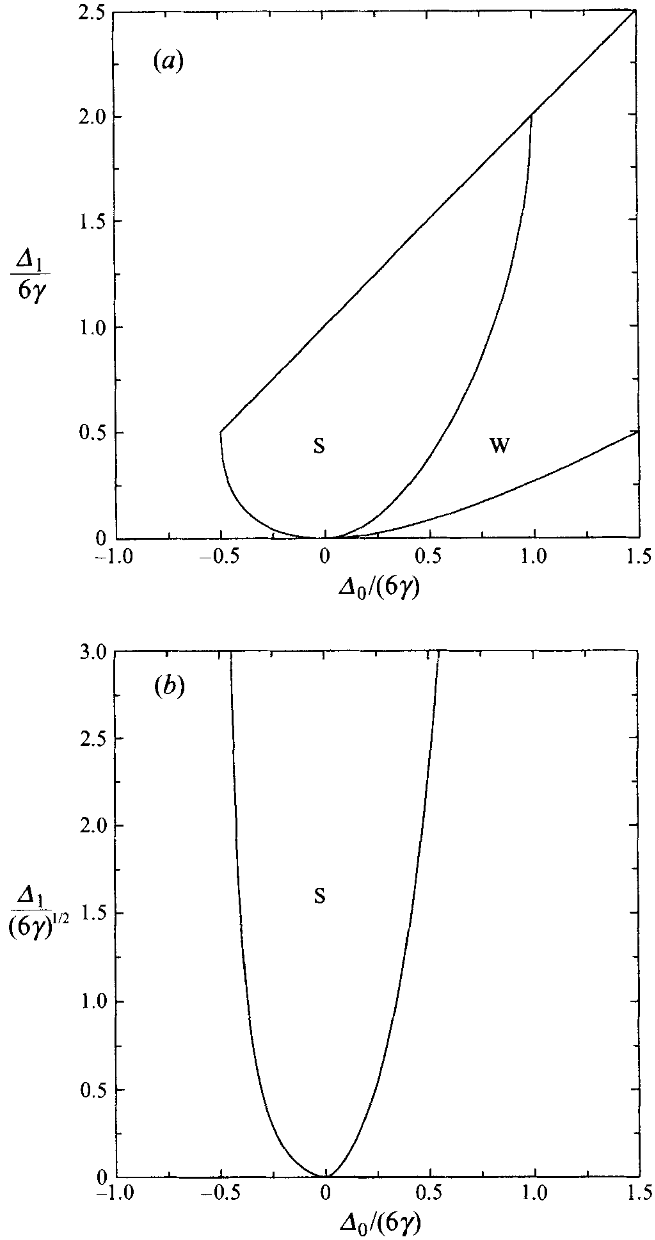

FIGURE 1. The resonant regimes for the solution of (1.2) with positive forcing. The abscissa is the normalized detuning parameter, while the ordinate is the normalized strength of the forcing. In physical terms these are proportional to the difference in the velocity of the fluid away from the contraction and the long-wave speed, and the magnitude of the contraction respectively. (*a*) Long contractions, (*b*) short contractions. S denotes the strong resonant regime and W the weak resonant regime.

These conditions can be used to determine regimes for the long and short contractions. For long contractions (4.5), together with the condition $\Delta_m \geqslant -6\gamma$, can be used to show that the strong and weak regimes are respectively

$$\left.\begin{array}{ll}\text{S:} & \begin{cases}6\gamma+\Delta_0-(36\gamma^2+12\Delta_0\gamma)^{1/2} < \Delta_1 < 6\gamma+\Delta_0, & -3\gamma < \Delta_0 < 0,\\ 6\gamma+\Delta_0-(36\gamma^2+12\Delta_0\gamma-3\Delta_0^2)^{1/2} < \Delta_1 < 6\gamma+\Delta_0, & 0 < \Delta_0 < 6\gamma,\end{cases}\\ \text{W:} & \begin{cases}6\gamma+\Delta_0-(36\gamma^2+12\Delta_0\gamma)^{1/2} < \Delta_1 < 6\gamma+\Delta_0-(36\gamma^2+12\Delta_0\gamma-3\Delta_0^2)^{1/2}, & \\ & 0<\Delta_0<6\gamma,\\ 6\gamma+\Delta_0-(36\gamma^2+12\Delta_0\gamma)^{1/2} < \Delta_1 < 6\gamma+\Delta_0, & \Delta_0 > 6\gamma.\end{cases}\end{array}\right\} \quad (4.33)$$

For short contractions the only distinct critical regime is the strong resonant regime, the lee-wave regime now has no lower limit on the subcritical domain. This can be demonstrated from (4.26) by noting that $A_+ = 0$ implies $\Delta_1 = 0$. Therefore, for short contractions the resonant regime is

$$\text{S:} \quad \begin{cases} D\Delta_1 > \dfrac{2(-\Delta_0)^{3/2}}{\surd 3(6\gamma+2\Delta_0)}, & -3\gamma < \Delta_0 < 0,\\[2ex] D\Delta_1 > \dfrac{2(2\Delta_0)^{3/2}}{\surd 3(6\gamma-\Delta_0)}, & 0 < \Delta_0 < 6\gamma,\end{cases} \quad (4.34)$$

These regimes are shown in figure 1. Together, these results for short and long contractions demonstrate that strong resonance can only occur if Δ_0 satisfies

$$-3\gamma < \Delta_0 < 6\gamma. \quad (4.35)$$

5. Numerical solutions for positive forcing

In the next two sections numerical solutions of (1.2) are presented for positive and negative forcing. These are obtained using the pseudospectral method of Fornberg & Whitham (1978). In both sections the shape of the contraction is described by

$$\theta(x) = \operatorname{sech}^2 x. \quad (5.1)$$

Figure 2(*a*) shows a typical example of the behaviour in the strong resonant regime, demonstrating all the features outlined in the previous section. First, a steady solution forms within the contraction, with a depression extending downstream. A continuous train of waves is generated at the upstream end of the contraction, which have form close to that of solitary waves. The depression downstream of the contraction is terminated by a receding wavetrain, for which the variation of the modulus can be clearly seen. At the leading edge of this wavetrain the waves are slowly separating, indicating that they are forming into solitary waves with $m = 1$. At the trailing end the amplitude of the waves approaches zero and they have a marked dispersive character, indicative that $m \rightarrow 0$.

An example of the behaviour for the weak resonant or lee-wave regime is shown in figure 2(*b*). The main features to notice are the wavetrain upstream of the contraction terminated by a mean level, which then forms a steady solution within the contraction and a lee wavetrain downstream of the contraction. The region that the lee wavetrain occupies can be seen to be linearly increasing with time, as also does the region that the

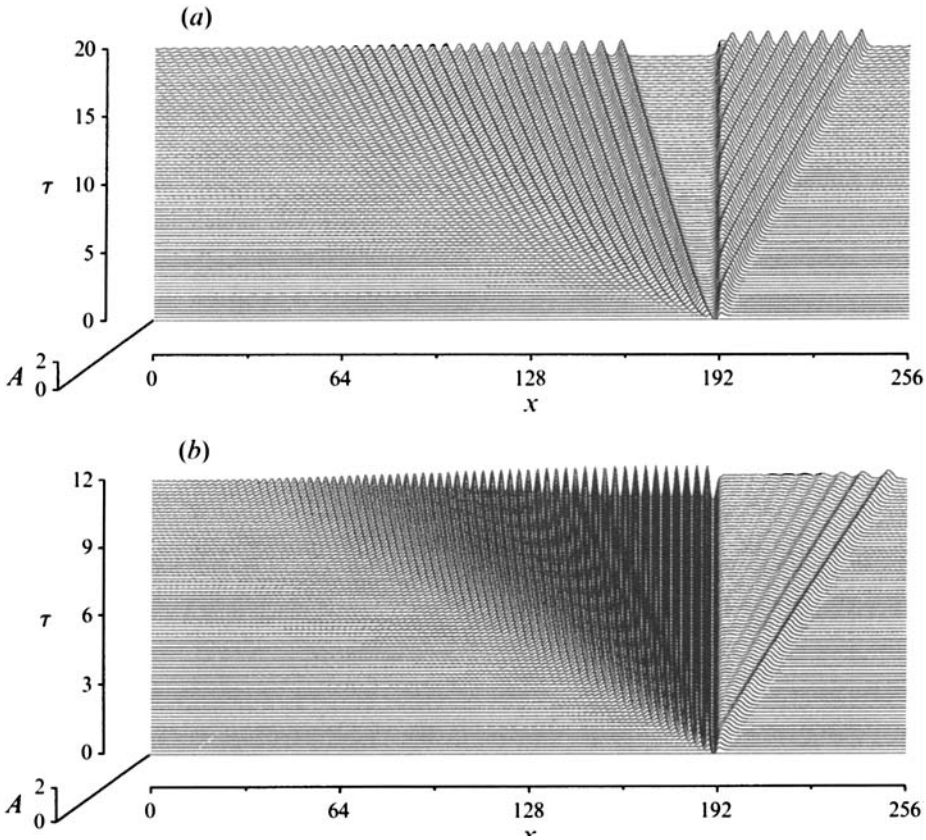

FIGURE 2. Numerical solutions of (1.2) for $x_a = 1$, $\Delta_1 = 2$ and $\gamma = 1$. (a) $\Delta_0 = 0$, (b) $\Delta_0 = 4$.

downstream wavetrain occupies. Again, downstream of the lee wavetrain there is a partial wavetrain with $m \to 0$ at the trailing edge.

The analytical approximations demonstrate that the variation in the upstream and downstream amplitude of the inner solution governs the behaviour of the waves that propagate away from the contraction. Therefore, in figure 3 the numerical values for these steady amplitudes are compared to the analytical predictions. In figure 3(a) comparisons are made for the strong resonant regime, and as can be seen there is very good agreement between the numerical and analytical solutions. In figure 3(b) the comparison is for the weak resonant regime and again the agreement is very good. One point to note from this figure is that as $x_a \to 0$ the analytical results show that $A_+ \to 0$. Therefore, in this limit the lee-wave regime encompasses the whole subcritical domain.

Both of these figures show a monotonic increase in the upstream and downstream height of the inner solution with the width of the contraction. However, this is not always the case, especially if it arises that the long- and short-contraction limits fall into different regimes. One example is if the long-contraction limit of upstream height, A_+, is small. In this case as x_a increases A_+ will increase from zero to a maximum value and then approach the long contraction limit.

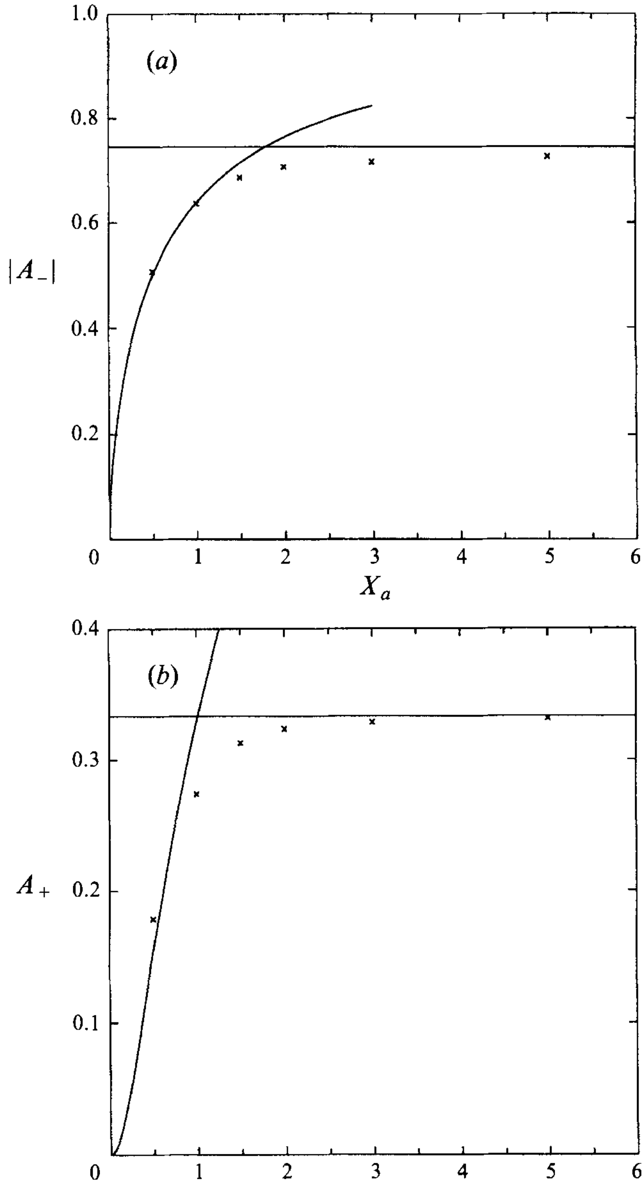

FIGURE 3. (*a*) The downstream depression height A_- versus the length of the forcing x_a for solutions of (1.2) with $\Delta_0 = 0$, $\Delta_1 = 2$ and $\gamma = 1$. The solid lines denote the analytical limits, while the crosses are obtained from numerical solutions. (*b*) As in (*a*) but now the upstream height A_+ is shown and $\Delta_0 = 4$.

6. Numerical solutions for negative forcing

Although no analytical solutions can be constructed for negative forcing, we would expect that the regimes identified in §4 give some indication of the behaviour and, therefore, these are used as a starting point in classifying the numerical results.

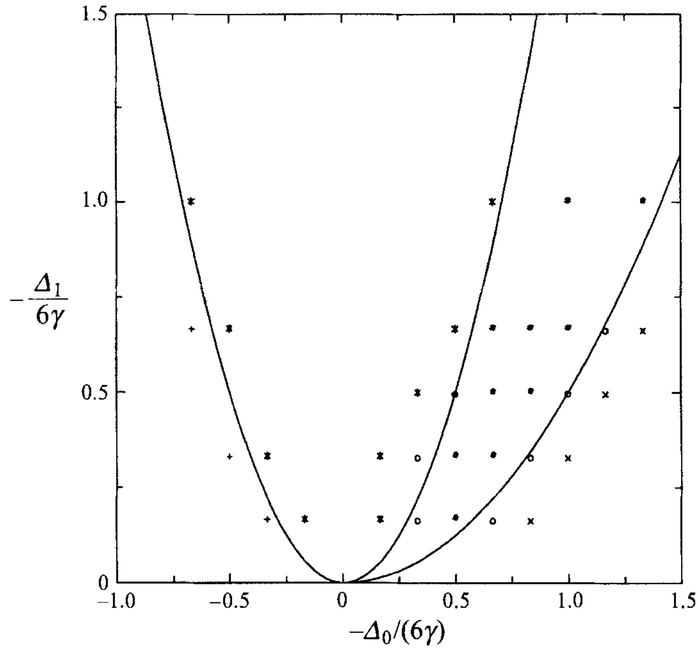

FIGURE 4. Classification of numerical solutions of (1.2) for negative forcing and $x_a = 3$: +, supercritical; $*$, strong resonant; #, weak resonant; ×, subcritical; ○, transitional cases. The curves separating the various regimes are given by (6.1). As in figure 1 the abscissa is the normalized detuning parameter, while the ordinate is the normalized strength of the forcing.

6.1. *Long contractions*

In this limit, the boundaries of the various regimes should be functions of Δ_0/γ and Δ_1/γ. Therefore, in figure 4 numerical solutions of (1.2) are classified in terms of these two parameters. Again four regimes are found. For $|\Delta_0| \gg \Delta_1$ supercritical and subcritical behaviour occurs, where a stationary disturbance develops in the contraction and either upstream or downstream of the contraction is a freely propagating transient dispersive wave. Between the subcritical and supercritical regimes there exists two distinct resonant regimes, which will again be termed strong and weak. From the classification of the numerical results it appears that the limits of these regimes are

$$\left.\begin{aligned} &\text{S:} \quad -\Delta_1/6\gamma > 2(\Delta_0/6\gamma)^2, \\ &\text{W:} \quad \tfrac{1}{2}(\Delta_0/6\gamma)^2 < -\Delta_1/6\gamma < 2(\Delta_0/6\gamma)^2, \quad \Delta_0 > 0. \end{aligned}\right\} \tag{6.1}$$

In figure 5(*a*) an example of the strong resonant regime is shown, for which the main characteristics of this regime are apparent. Initially a positive disturbance forms downstream of the throat of the contraction and a negative disturbance upstream, in accordance with the small-time solution $A \sim \gamma\Delta_x\tau$. The negative disturbance develops into a stationary rarefaction, while the positive disturbance develops into a solitary wave and a wavetrain which propagates downstream. The solitary wave propagates upstream, being amplified on the downstream side of the contraction and damped on the upstream side. If it has sufficient initial energy it passes through the contraction and propagates upstream unchanged; one example of this can be seen. Otherwise it simply decays in the contraction. Once the wave has propagated upstream a new disturbance develops in the contraction. As nonlinear effects become important this develops into

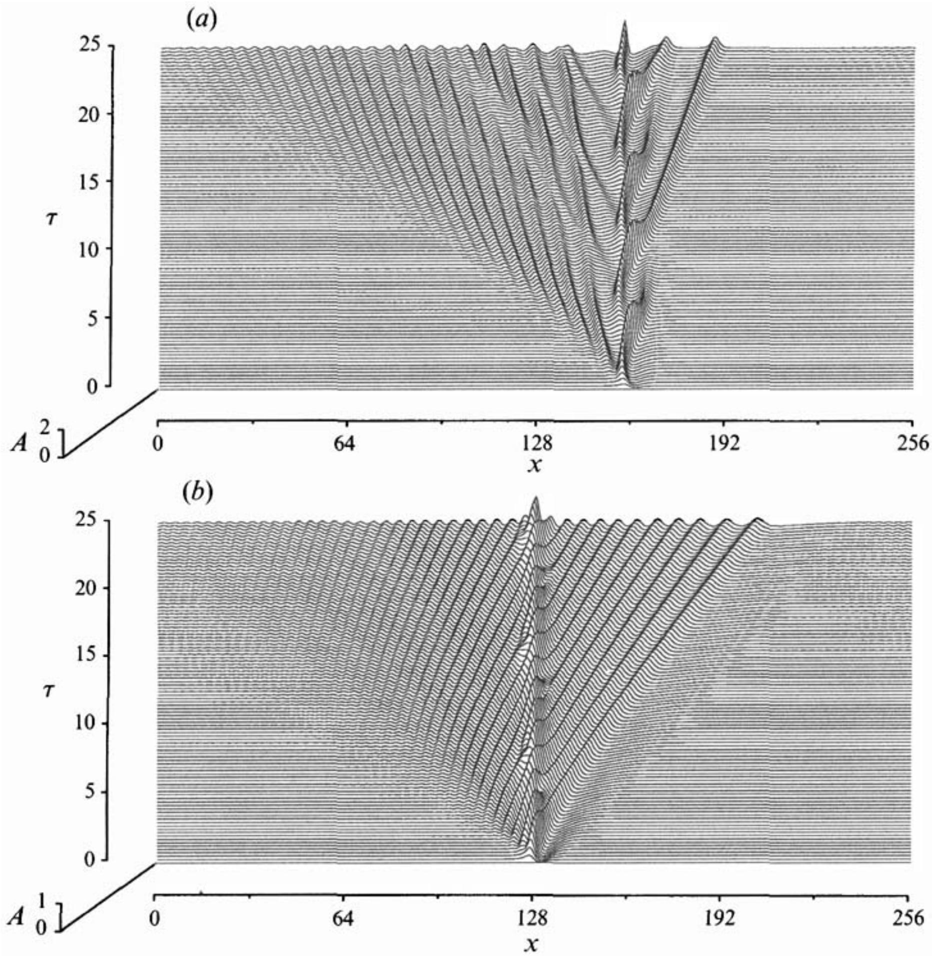

FIGURE 5. Numerical solutions of (1.2) for $\gamma = -1$, $x_a = 3$ and $\Delta_1 = 2$. (*a*) $\Delta_0 = 0$, (*b*) $\Delta_0 = 4$.

a large-amplitude solitary wave, which propagates upstream, and instead of a wavetrain, a small-amplitude long solitary wave which propagates downstream. The nonlinear interactions of this small-amplitude wave with the wavetrain can be clearly seen. A quasi-steady state will evolve in the contraction with large-amplitude solitary waves being radiated upstream and compensatory small-amplitude waves being radiated downstream.

A typical example of the weak resonant regime is shown in figure 5(*b*). Here the initial development is similar to that for the strong resonant regime. However, in this case the rarefaction is able to propagate upstream and the positive disturbance evolves into a wavetrain. These waves all have positive velocity and are able to propagate through the contraction. Within the contraction a stationary disturbance develops upon which these waves are superimposed. Therefore, at long times it would again be expected that a quasi-steady state forms, composed of a steady disturbance and superimposed propagating waves. Upstream of the contraction the asymptotic behaviour will be similar to the evolution of an initial well demonstrated by Fornberg & Whitham (1978).

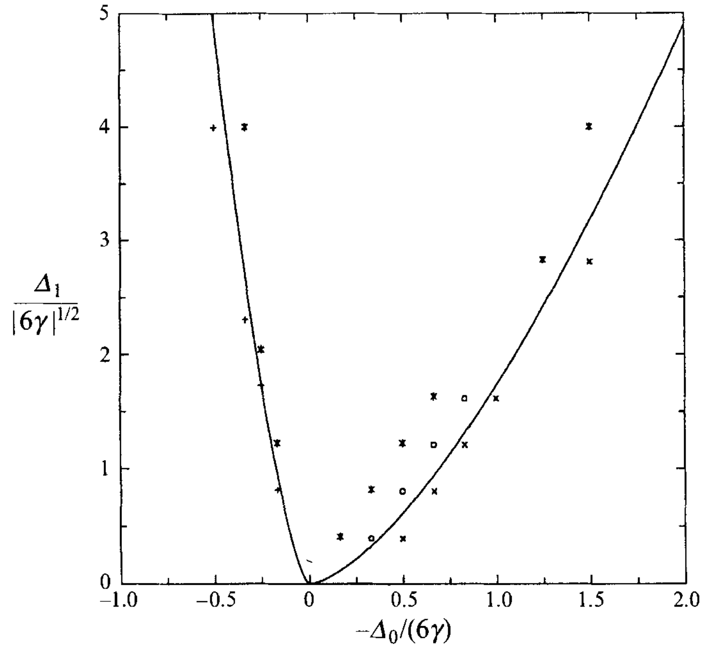

FIGURE 6. As for figure 4, except that in this case $x_a = 0.5$, there is no weak resonant regime and the curves are given by (6.2).

6.2. *Short contractions*

For short contractions the boundaries of the regimes would be expected to be functions of Δ_0/γ and $D\Delta_1|\gamma|^{-1/2}$. The numerical simulations were classified on this basis and the results are shown in figure 6. From §§4, 5 it was shown that for short contractions the weak resonant regime encompassed the whole subcritical domain. Similarly, there only appears to be three distinct regimes for negative forcing: supercritical, resonant and subcritical. The limits of the resonant regime appear to be given by

$$\text{S:} \quad \begin{cases} D\Delta_1 > -8\Delta_0/(3|2\gamma|^{1/2}), & \Delta_0 < 0, \\ D\Delta_1 > \Delta_0/(3|2\gamma|^{1/2}), & \Delta_0 > 0. \end{cases} \tag{6.2}$$

The example of the resonant behaviour shown in figure 7(*a*) illustrates the main characteristics of this regime. An initial disturbance forms which evolves into a stationary disturbance within the contraction, with a depression downstream. This depression is terminated by a wavetrain which propagates downstream. A train of large-amplitude waves is radiated upstream of the contraction and, as with the long contractions, for each of these large-amplitude waves a small-amplitude solitary wave is radiated downstream on the depression. Again the nonlinear interactions between these solitary waves and the wavetrain can be observed. In contrast to long contractions, the large-amplitude waves form and immediately propagate upstream without any decay or amplification. At large times a quasi-steady state will form within the contraction, with large-amplitude waves being periodically radiated upstream and compensatory small-amplitude waves being radiated downstream on the depression.

Figure 7(*b*) demonstrates the characteristics of the lee-wave regime, which are very similar to those for positive forcing. A lee wavetrain forms downstream of the contraction, which increases in length with time and is terminated by a partial

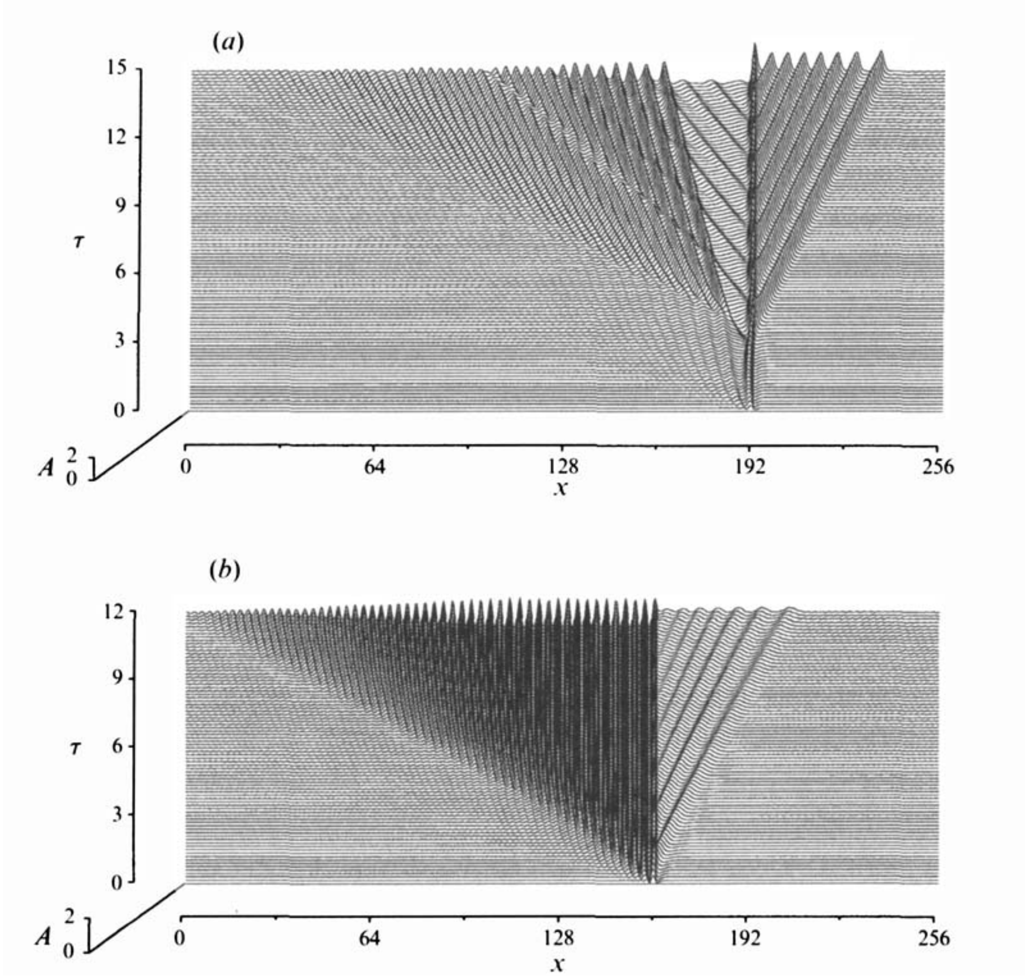

FIGURE 7. Numerical solution of (1.2) with $\gamma = -1$, $x_a = 0.5$, and $\Delta_1 = 2$. (*a*) $\Delta_0 = 0$, (*b*) $\Delta_0 = 4$.

wavetrain. Upstream of the contraction a train of solitary waves forms behind a rarefaction, similar to the behaviour shown in figure 5(*b*).

7. Discussion

The emphasis here has been on investigating the near-resonant generation of internal waves by the flow of a stratified fluid through a contraction. The response of the fluid in this circumstance can be categorized in terms of the parameter ϵ, being the perturbation in the width of the contraction, and the Boussinesq parameter β, defined by (2.2). First, if either the velocity of the oncoming flow or the width of the contraction vary with height, then the problem is analogous to stratified flow over a sill, where (1.1) applies and the response of the fluid is $O(\epsilon^{1/2})$. If the flow does not satisfy these conditions, then the resonant generation of waves is due to non-Boussinesq effects, and (1.2) applies. In this case the response will fall into one of two further categories: for arbitrary stratifications the wave amplitude is $O((\beta\epsilon)^{1/2})$, while if the fluid is approximately uniformly stratified, the wave amplitude is once again $O(\epsilon^{1/2})$.

An analysis of (1.2) has been undertaken here for the limits of long and short contractions. For positive forcing it has been shown that in the resonant regime the flow is steady within the contraction and at the upstream and downstream limits of the contraction waves are continually generated which then propagate away from the contraction. On the other hand, for negative forcing the flow is transient within the contraction in the resonant regime and waves are generated downstream of the contraction, which then propagate into the contraction and either decay or propagate upstream. Therefore, no steady state evolves in this case. This behaviour for negative and positive forcing broadly agrees with that found by Grimshaw & Smyth (1986) for (1.1).

The implications of the results found here for practical situations can be considered using as examples the first mode of a two-layer fluid, an exponentially stratified fluid and a linearly stratified fluid, all with a rigid lid (i.e. $e = 0$). For simplicity we will only consider the strong resonant regime and only the result for long contractions. First, the limits of the resonant regimes for (1.1) must be defined. If the maximum absolute value of G is denoted G_0, then Grimshaw & Smyth (1986) demonstrated that for positive forcing, where $G_0 > 0$, the strong resonant regime is given by

$$-\tfrac{1}{2}(12G_0)^{1/2} \leqslant \varDelta \leqslant (12G_0)^{1/2}, \tag{7.1}$$

while for negative forcing, where $G_0 < 0$, the resonant regime is

$$|\varDelta| \leqslant (12|G_0|)^{1/2}. \tag{7.2}$$

These results of Grimshaw & Smyth (1986) are applicable to a two-layer fluid with the thermocline at height z_0, flowing through a contraction where the width increases linearly with depth. The width of the contraction is in this case

$$b_{\pm}(x,z) = \pm b_0(1-\epsilon(1-z)\,g(x)), \tag{7.3}$$

where $\max(g) = 1$. In the Boussinesq limit the velocity of long waves is $c = (z_0(1-z_0))^{1/2}$, and

$$G_0 = (3-2z_0)(2z_0-1)/(8c) \tag{7.4}$$

Therefore if $z_0 > \frac{1}{2}$ the forcing is positive and the strong resonant regime is given by

$$-\tfrac{3}{2}z_0(1-z_0)(\tfrac{3}{2}\epsilon(3-2z_0)(2z_0-1))^{1/2} \leqslant \bar{u}+c \leqslant 3z_0(1-z_0)(\tfrac{3}{2}\epsilon(3-2z_0)(2z_0-1))^{1/2}. \tag{7.5}$$

If $z_0 < \frac{1}{2}$ the forcing is negative and the strong resonant regime is

$$|\bar{u}+c| \leqslant 3z_0(1-z_0)(\tfrac{3}{2}\epsilon(3-2z_0)(1-2z_0))^{1/2}. \tag{7.6}$$

If the width of the contraction is constant with height, then in the resonant bands the waves will have magnitude $O((\beta\epsilon)^{1/2})$ and (1.2) applies. When $z_0 < \frac{1}{2}$ the forcing is now positive, and the resonant band is

$$-\tfrac{3}{4}\beta(1-2z_0)\,z_0^2(1-z_0)^2 < \bar{u}+c < \tfrac{3}{2}\beta(1-2z_0)\,z_0^2(1-z_0)^2. \tag{7.7}$$

While if $z_0 > \frac{1}{2}$, the more common case, the forcing is negative, thus the resonant band is

$$|\bar{u}+c| < \tfrac{1}{4}(\tfrac{1}{3}\beta\epsilon(2z_0-1)\,z_0^3(1-z_0)^2)^{1/2}. \tag{7.8}$$

For exponential and linear stratification we only consider here the resonant bands in a straight-sided contraction. Again (1.2) applies, and in the resonant band the waves have magnitude $O(\epsilon^{1/2})$. In both cases the wave speed of the first-mode case will be $c_0 = \pi^{-1}$. It can be shown for exponential stratification that the forcing is positive, therefore the resonant band is given by

$$-\tfrac{2}{3}\beta^2c_0^5 < \bar{u}+c_0 < \tfrac{4}{3}\beta^2c_0^5. \tag{7.9}$$

For linear stratification the forcing is negative, consequently the resonant band is

$$|\bar{u}+c_0| < (2\epsilon)^{1/2}\beta c_0^3. \tag{7.10}$$

Note that for both linear and exponential stratification, although waves of significant amplitude are generated, the resonant bands are very small.

The analysis undertaken here is strictly only applicable to a perturbation in an otherwise uniform channel; however, it does provide a first step to understanding what occurs at resonance. One consequence is that at or near resonance, and, as here, in the absence of friction, the flow in the vicinity of the contraction remains unsteady, with internal waves being generated both upstream and downstream of the contraction. For positive forcing localized regions of steady flow can occur at the contraction, and in the immediate downstream region. In this case, these represent a possible path to localized steady hydraulic solutions via the resonant generation of internal waves and the permanent modification of the initially undisturbed flow. On the other hand, for negative forcing the flow remains unsteady at the contraction, with waves being generated downstream and then propagating upstream. Whether such conclusions apply when the weakly nonlinear assumption breaks down is a point which requires further investigation.

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